Energetic homogenization of one-dimensional porous material with variable cross-sectional area

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A recently developed energetic homogenization method [Muhlestein, J. Acoust. Soc. Am. ***5****, 3584-3593 (2020)] is used to estimate the apparent material properties of a simple porous material with varying cross-sectional areas. Expressions for the apparent mass density, compressibility, and kinematic viscosity are given. As a demonstration, a simple duct with identical expansion chambers is briefly considered.*

1 Introduction

This paper is concerned with the homogenization of a porous material that may be modeled as a statistically identical network of one-dimensional ducts with varying cross-sectional areas. For this paper the ducts are considered to be acoustically wide, such that boundary layers may be assumed to be thin compared to the duct width. Using a recently developed energetic homogenization technique, described below in Sec. 2, the impact of the varying cross-sectional area on the apparent mass density, bulk modulus, and kinematic viscosity may be calculated. A demonstration of the technique using a simple expansion chamber is given in Sec. 3.

2 Energetic Homogenization

One approach to homogenization is to average the weak form of the dynamic equations, known as Hamilton's principle, by assuming that the acoustic pressure and volume velocity are constants throughout a representative volume element [1]. Hamilton's principle requires knowledge of the total kinetic and potential energies, as well as any constraints and external forces (such as from friction). For a one-dimensional duct with variable cross-section we may write the kinetic and potential energies, respectively, as

$$E_k = \int_0^L \frac{1}{2S(x)} \rho_0 q^2 \, dx, \qquad E_p = \int_0^L \frac{S(x)}{2} \beta_0 p^2 \, dx, \tag{1}$$

where the domain $x \in (0, L)$ is the extent of the duct of interest, $S(x)$ is the cross-sectional area, ρ_0 is the fluid mass density, β_0 is the fluid compressibility (inverse bulk modulus), q is the volume velocity, and p is the acoustic pressure. The acoustical fields are assumed to be functions of position x and time t. The continuity equation may be imposed via the constraint equation $\mathcal{C} = 0$ and boundary layer losses may be incorporated as an external force density F [2], where

$$\mathcal{C} \equiv S(x)\beta_0 \frac{\partial p}{\partial t} + \frac{\partial q}{\partial x} = 0, \qquad F = -\rho_0 \sqrt{\nu_0} \frac{C(x)}{S^2(x)} \frac{\partial^{1/2} q}{\partial t^{1/2}}, \tag{2}$$

where ν_0 is the kinematic viscosity that depends only on the propagation fluid, $C(x)$ is the duct perimeter at position x, and $\partial^{1/2}/\partial t^{1/2}$ is the half time derivative operator (see, e.g., Ref. [3]). Combining these expressions, we may write Hamilton's principle as

$$\delta \int_{t_0}^{t_1} \int_0^L \left[\frac{\rho_0}{2S(x)} q^2 - \frac{S(x)\beta_0}{2} p^2 + \mu \mathcal{C} \right] dx \, dt = \int_{t_0}^{t_1} \int_0^L \rho_0 \sqrt{\nu_0} \frac{C(x)}{S(x)} \frac{\partial^{1/2} q}{\partial t^{1/2}} \delta u \, dx \, dt, \tag{3}$$

where t_0 and t_1 are arbitrary initial and final times, δ is the variational operator, μ is a Lagrange multiplier function for the constraint equation, and u is the volume displacement (such that $q = \partial u/\partial t$). Note that in this form of Hamilton's principle the only acoustical field quantities that appear are those that are continuous across interfaces (i.e., the acoustic pressure and various time derivatives or integrals of the volume displacement). Then, following Muhlestein [1], we may conclude the macroscopic acoustical quantities behave as though the duct had constant (and chosen arbitrarily) cross-sectional

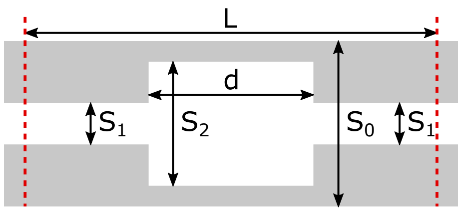

Figure 1: Schematic of an example duct with a variable cross-sectional area, as well as the reference cross-sectional area S_0.

area S_0 and perimeter C_0, and the propagation fluid had effective mass density, compressibility, and boundary drag coefficient respectively given by

$$\rho_{\text{eff}} = \rho_0 S_0 \left\langle \frac{1}{S} \right\rangle, \qquad \beta_{\text{eff}} = \beta_0 \frac{\langle S \rangle}{S_0}, \qquad \rho_{\text{eff}} \sqrt{\nu_{\text{eff}}} = \rho_0 \sqrt{\nu_0} \frac{S_0}{C_0} \left\langle \frac{C}{S} \right\rangle. \tag{4}$$

From these quantities one may derive an effective wavenumber, acoustic impedance, and absorption values.

While this presentation is limited to considering boundary layer losses in linear media, bulk thermoviscous losses and nonlinear phenomena may be readily incorporated (see Ref. [1]). Additional geometric features (e.g., dead-end pores) or physical phenomena may also be accounted for using this formalism.

3 Demonstration

As a demonstration of the homogenization method, consider the case of an air-filled duct with simple cylindrical expansion chambers (see Fig. 1). Let the reference cross-sectional area be S_0, the cross-sectional area of the narrow tubes be S_1, and the cross-sectional area of the expansion chambers be S_2, the length of the element is L and the length of the expansion chamber is d. Using Eqs. (4) we may then write

$$\frac{\rho_{\text{eff}}}{\rho_0} = \frac{S_0}{L} \frac{S_1 d + S_2(L-d)}{S_1 S_2}, \qquad \frac{\beta_{\text{eff}}}{\beta_0} = \frac{S_1(L-d) + S_2 d}{L S_0}, \qquad \sqrt{\frac{\nu_{\text{eff}}}{\nu_0}} = \frac{C_2 S_1 d + C_1 S_2 (L-d)}{C_0 S_1 d + C_0 S_2 (L-d)}, \tag{5}$$

where C_i is the circumference associated with S_i. Since the porosity may be written as $\phi = [S_1(L - d) + S_2 d]/L$, the structure factor may be written as

$$k_s = \phi \frac{\rho_{\text{eff}}}{\rho_0} = \frac{[S_1(L-d) + S_2 d]\,[S_2(L-d) + S_1 d]}{L^2 S_1 S_2}, \tag{6}$$

in agreement with Neithalath, et al. [4].

It is possible that more extreme effective properties could be obtained by incorporating more complicated geometries, such as dead-end resonators or membranes.

References

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