Closed-forms derivation and experimental validation of the Willis coupling of different asymmetric structures

M. Malléjac [1], A. Merkel, [2], D. Torrent [3], J. Li [4], V. Tournat [1], V. Romero-García [1], and J.-P. Groby [1]

[1] Laboratoire dAcoustique de l' Université du Mans (LAUM), UMR 6613, Institut d'Acoustique - Graduate School (IA-GS), CNRS, Le Mans Université, France
[2] Université de Lorraine, CNRS, IJL, F-54000 Nancy, France
[3] GROC, UJI, Institut de Noves Tecnologies de la Imatge (INIT), Universitat Jaume I, 12071 Castelló, Spain
[4] Department of Physics, The Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, China

We derive analytically and verify experimentally the Willis coupling parameters of simple one-dimensional asymmetric and reciprocal resonant structures made of Helmholtz resonators and plates. The coupling terms of the derived analytical closed-forms are analyzed in each of the three configurations.

1 Introduction

Bi-isotropic media [1] and their acoustic counterparts, the Willis materials [2, 4, 3], have drawn considerable interest. These materials have the particularity to couple the potential and kinetic energy in their constitutive equations, therefore enhancing the possibility of control and design for desired applications. In this talk, we will derive and analyze the closed-forms of the effective and coupling parameters for three different asymmetric and reciprocal unit cells composed of two detuned Helmholtz resonators, two plates, or one Helmholtz resonator and one plate. The closed-forms will be analyzed to investigate the influence of the nature of the asymmetry on the coupling terms.

2 Derivation of the closed-forms

One dimensional asymmetric and reciprocal media composed of a periodic arrangement of d-thick unit cells are considered. The closed-forms of the effective parameters are derived from the transfer matrix of the unit cell $\mathbf{T}$ that links the state vector $\mathbf{W} = \langle p, v \rangle^t$ (pressure p and particle velocity v) from one side to the other of the unit cell

$$\mathbf{W}(\mathbf{d}) = \mathbf{T}\mathbf{W}(0) = e^{\mathbf{A}d}\mathbf{W}(0). \tag{1}$$

The $\mathbf{A}$ matrix arises from the constitutive equations and reads for a Willis material as

$$\mathbf{A} = i\omega \begin{bmatrix} \chi & \rho \\ 1/K & -\chi \end{bmatrix}, \tag{2}$$

where ρ is the density, K the bulk modulus, and χ the Willis coupling term.
Making use of the Padé's approximation of the matrix exponential,

$$\mathbf{T} = e^{\mathbf{A}d} \approx (\mathbf{Id} - \mathbf{A}d/2)^{-1} (\mathbf{Id} + \mathbf{A}d/2), \tag{3}$$

one can approximate the matrix $\mathbf{A}$ in terms of the t_{ij} elements of the $\mathbf{T}$ matrix

$$\mathbf{A} \approx \frac{2}{d(2 + t_{11} + t_{22})} \begin{bmatrix} t_{11} - t_{22} & 2t_{12} \\ 2t_{21} & t_{22} - t_{11} \end{bmatrix}, \tag{4}$$

by ensuring the reciprocity of the transformation.
We can then identify the closed-forms of the effective parameters and Willis coupling term

$$\chi = \frac{-2i(t_{11} - t_{22})}{\omega d(2 + t_{11} + t_{22})}, \tag{5}$$

$$\rho = \frac{-4it_{12}}{\omega d(2 + t_{11} + t_{22})}, \tag{6}$$

$$[K]^{-1} = \frac{-4it_{21}}{\omega d(2 + t_{11} + t_{22})}. \tag{7}$$

3 Experimental validation of the derived closed-forms

The closed-forms of the effective parameters for the three unit cells depicted in Fig. 1 are compared against numerical simulations and experimental measurements. The measurements are performed with a 4 microphones impedance tube of radius $r = 2.5$ cm excited by a stept sine.

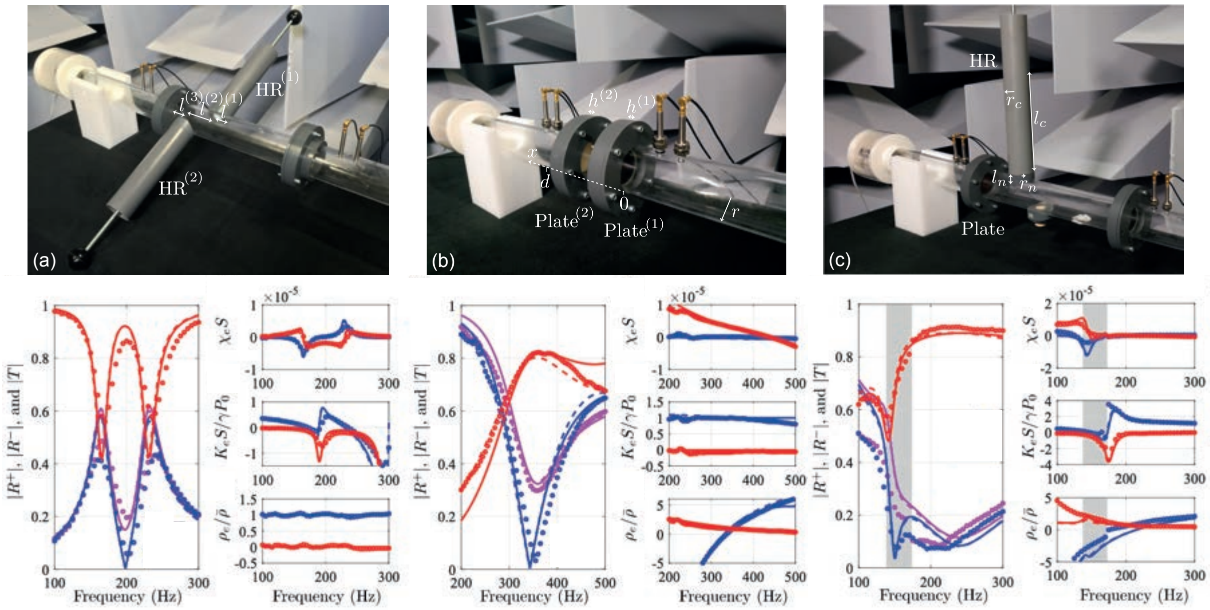

Figure 1: Photographs and results for the three cases analyzed: (a) two detuned Helmholtz resonators, (b) two detuned plates, and (c) plate and Helmoltz resonator. The different subplots show respectively the scattering magnitudes, the Willis coupling parameter, the normalized effective bulk modulus, and the normalized effective density, for each case. The solid lines represent the analytical, the dashed lines the numerical, and the symbols the experimental results.

A very good agreement is found between the three methods, thus validating the analytical derivation of the effective parameters and the Willis coupling.

4 Conclusion

This work shows a new approach for the understanding of the Willis metamaterials: the derivation of the closed-forms of the effective parameters and Willis coupling terms via the Padé's approximation of transfer matrix of different one dimensional asymmetric and reciprocal unit cells. We discuss the different types of coupling terms due to the nature of the asymmetry, either related to a detuning of identical type resonators or due to a physical asymmetry when the unit cell involves different types of resonators. A two-fold validation is performed with numerical and experimental results that matches well the derived closed-forms.

References

[1] I. Lindell, A. Sihvola, S. Tretyakov, and A. J. Viitanen. *Electromagnetic waves in chiral and bi-isotropic media*, pages 122. Artech House, 1994.

[2] J. R. Willis. *Variational principles for dynamic problems for inhomogeneous elastic media*, Wave Motion, 3:1 11, 1981.

[3] A. Merkel, V. Romero-García, J.-P. Groby, J. Li, and J. Christensen. *Unidirectional zero sonic reflection in passive PT-symmetric willis media*. Phys. Rev. B, 98:201102, 2018.

[4] M. B. Muhlestein, C. F. Sieck, P. S. Wilson, and M. R. Haberman. *Experimental evidence of Willis coupling in a one-dimensional effective material element*. Nat Commun, 8:15625, 2017.